SPECIAL FUNCTIONS and POLYNOMIALS

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Many of the special functions and polynomials are constructed along standard procedures In this short survey we list the most essential ones.

July 21, 2004

1 Legendre Polynomials $P_\ell(x)$.

Differential Equation:

$$(1-x^2)\,P_\ell{}''(x) - 2x\,P_\ell{}'(x) + \ell(\ell+1)\,P_\ell(x) \quad = \quad 0\ ,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\,P_\ell(x) + \ell(\ell+1)\,P_\ell(x) \quad = \quad 0\ . \tag{1.1}$$

Generating function:

$$\sum_{\ell=0}^{\infty} P_\ell(x)t^\ell \quad = \quad (1-2xt+t^2)^{-\frac{1}{2}} \qquad \text{for} \quad |t|<1, |x|\le 1. \tag{1.2}$$

Orthonormality:

$$\int_{-1}^{1} P_\ell(x)\,P_{\ell'}(x)\,\mathrm{d}x \quad = \quad \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\ , \tag{1.3}$$

$$\sum_{\ell=0}^{\infty} P_\ell(x)P_\ell(x')(2\ell+1) \quad = \quad 2\delta(x-x')\ . \tag{1.4}$$

Expressions for $P_\ell(x)$:

$$P_\ell(x) \quad = \quad \frac{1}{2^\ell}\sum_{\nu=0}^{[\ell/2]} \frac{(-1)^\nu\,(2\ell-2\nu)!}{\nu!\,(\ell-\nu)!\,(\ell-2\nu)!}\,x^{\ell-2\nu} \tag{1.5}$$

$$= \quad \frac{1}{\ell!\,2^\ell}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^\ell\,(x^2-1)^\ell\ , \tag{1.6}$$

$$= \quad \frac{1}{\pi}\int_0^\pi (x+\sqrt{x^2-1}\cos\varphi)^\ell\,\mathrm{d}\varphi\,. \tag{1.7}$$

Recurrence relations:

$$\ell\,P_{\ell-1} - (2\ell+1)\,x\,P_\ell + (\ell+1)\,P_{\ell+1} = 0\ ;$$

$$P_\ell = x\,P_{\ell-1} + \frac{x^2-1}{\ell}\,P'_{\ell-1} \quad ;$$

$$xP'_\ell \;-\; \ell\,P_\ell = P'_{\ell-1} \quad ;$$

$$xP'_\ell + (\ell+1)\,P_\ell \quad = \quad P'_{\ell+1} \quad ;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[P_{\ell+1} - P_{\ell-1}] = (2\ell+1)\,P_\ell. \tag{1.8}$$

Examples:

$$P_0 = 1\ , \qquad P_1 = x\ , \qquad P_2 = \tfrac{1}{2}(3x^2-1)\ , \qquad P_3 = \tfrac{1}{2}x(5x^2-3)\ . \tag{1.9}$$

2 Associated Legendre Functions $P_\ell^m(x)$.

Differential equation:

$$(1-x^2)\,P_\ell^m(x)\,''-2x\,P_\ell^m(x)\,'+\Big(\ell(\ell+1)-\frac{m^2}{1-x^2}\Big)\,P_\ell^m(x)\;=\;0\;. \tag{2.1}$$

Generating function:

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{P_\ell^m(x)\,z^m\,y^\ell}{m!}\;=\;\Big[1-2y\Big(x+z\sqrt{1-x^2}\Big)+y^2\Big]^{-\frac{1}{2}}\;. \tag{2.2}$$

Orthogonality:

$$\int_{-1}^{1}P_\ell^m(x)\,P_{\ell'}^m(x)\,\mathrm{d}x=\frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}\;,\qquad(\,\ell,\,\ell'\geq m\,)\;. \tag{2.3}$$

$$\sum_{\ell=m}^{\infty}(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}\,P_\ell^m(x)\,P_\ell^m(x')=2\delta(x-x')\;,\qquad(\,|x|<1\text{ and }|x'|<1\,)\;. \tag{2.4}$$

Expressions for $P_\ell^m(x)$[1]:

$$P_\ell^m(x)\;=\;(1-x^2)^{\frac{1}{2}m}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_\ell(x)\;. \tag{2.5}$$

$$P_\ell^m(x)\;=\;\frac{(\ell+m)!}{\ell!\,\pi}(-1)^{m/2}\int_0^\pi\Big(x+\sqrt{x^2-1}\cos\varphi\Big)^\ell\cos m\varphi\,\mathrm{d}\varphi\;. \tag{2.6}$$

Recurrence relations:

$$P_\ell^{m+1}-\frac{2mx}{\sqrt{1-x^2}}P_\ell^m+\{\ell(\ell+1)-m(m-1)\}P_\ell^{m-1}=0 \tag{2.7}$$

$$\sqrt{1-x^2}\,P_\ell^{m+1}(x)\;=\;(1-x^2)\,P_\ell^m(x)\,'+mx\,P_\ell^m(x)\;,$$

$$(2\ell+1)x\,P_\ell^m=(\ell+m)\,P_{\ell-1}^m+(\ell+1-m)\,P_{\ell+1}^m\;, \tag{2.8}$$

$$x\,P_\ell^m\;=\;P_{\ell-1}^m-(\ell+1-m)\sqrt{1-x^2}\,P_\ell^{m-1}\;,$$

$$P_{\ell+1}^m-P_{\ell-1}^m\;=\;(2\ell+1)\,P_\ell^{m-1}\sqrt{1-x^2}\;, \tag{2.9}$$

and various others.

Examples:

$$\begin{aligned}P_1^1&=\sqrt{1-x^2}\;, &\qquad P_2^2&=3(1-x^2)\;,\\ P_2^1&=3x\sqrt{1-x^2}\;, &\qquad P_3^2&=15\,x(1-x^2)\;.\end{aligned} \tag{2.10}$$

[1]Note that some authors define $P_\ell^m(x)$ with a factor $(-1)^m$, giving $P_\ell^m(x)=(-1)^m(1-x^2)^{\frac{1}{2}m}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_\ell(x)$. Obviously this minus sign propagates to the generating function, the recurrence relations and the explicit examples, when m is odd.

3 Bessel $J_n(x)$ and Hankel $H_n(x)$ functions.

Differential equation (for both J_n and H_n):

$$x^2\, J_n''(x) + x\, J_n'(x) + (x^2 - n^2)\, J_n(x) \;=\; 0\;. \tag{3.1}$$

Generating function (if n integer):

$$\sum_{n=-\infty}^{\infty} J_n(\alpha x) \left(\frac{s}{\alpha}\right)^n \;=\; e^{\frac{x}{2}\left(s - \frac{\alpha^2}{s}\right)}\;, \tag{3.2}$$

$$J_{-n} = (-1)^n\, J_n\;.$$

Orthogonality:

$$\int_0^\infty \xi\, J_n(\alpha\xi)\, J_n(\beta\xi)\, \mathrm{d}\xi = \frac{1}{\alpha}\, \delta(|\alpha| - |\beta|)\;. \tag{3.3}$$

$$\int_0^a \xi\, J_n(\alpha\xi)\, J_n(\beta\xi)\, \mathrm{d}\xi = \frac{a^2}{2}\, \{J_{n+1}(\alpha a)\}^2 \delta_{\alpha\beta}\;. \tag{3.4}$$

if in the 2^{nd} relation α, β are roots of the equation $J_n(\alpha\xi) = 0$.

Expressions for $J_n(x)$ (for n integer):

$$J_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{n+2k} \;=\; \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \oint t^{-n-1} \mathrm{d}t\, e^{\,t - x^2/4t}\;. \tag{3.5}$$

$$J_n(x) \;=\; \frac{1}{\pi} \int_0^\pi \cos\,(n\theta - x\,\sin\theta)\, \mathrm{d}\theta\;. \tag{3.6}$$

Recurrence relations (for both J_n and H_n):

$$\frac{d}{dx}\{x^n\, J_n(x)\} \;=\; x^n\, J_{n-1}(x)\;;$$

$$J_{n-1}(x) + J_{n+1}(x) \;=\; \frac{2n}{x}\, J_n(x)\;;$$

$$J'_n(x) \;=\; J_{n-1}(x) - \frac{n}{x}\, J_n(x) \;=\; \frac{n}{x}\, J_n(x) - J_{n+1}(x) = \tfrac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \tag{3.7}$$

Relations between Hankel and Bessel functions:

$$H_n^{(1)}(x) \;=\; \frac{i}{\sin n\pi} \left(e^{-n\pi i} J_n(x) - J_{-n}(x)\right)\;;$$

$$H_n^{(2)}(x) \;=\; \frac{-i}{\sin n\pi} \left(e^{n\pi i} J_n(x) - J_{-n}(x)\right)\;, \tag{3.8}$$

so that

$$J_n(x) \;=\; \tfrac{1}{2}\left(H_n^{(1)}(x) + H_n^{(2)}(x)\right)\;;$$

$$J_{-n}(x) \;=\; \tfrac{1}{2}\left(e^{n\pi i} H_n^{(1)}(x) + e^{-n\pi i} H_n^{(2)}(x)\right)\;. \tag{3.9}$$

4 Spherical Bessel Functions $j_\ell(x)$.

Differential equation:

$$(xj_\ell)'' + \left(x - \frac{\ell(\ell+1)}{x}\right) j_\ell = 0 \ . \tag{4.1}$$

Generating Function:

$$\sum_{\ell=0}^{\infty} \frac{j_\ell(x)\, t^\ell}{\ell!} = j_0\left(\sqrt{x^2 - 2xt}\right) \ . \tag{4.2}$$

Orthogonality:

$$\int_0^\infty x^2 j_\ell(\alpha x)\ j_\ell(\beta x)\ \mathrm{d}x = \frac{\pi}{2\alpha}\, \delta(\alpha - \beta) \ . \tag{4.3}$$

$$\int_{-\infty}^\infty j_\ell(x)\, j'_\ell(x)\ \mathrm{d}x = \frac{\pi}{2\ell + 1}\, \delta_{\ell\ell'} \ . \tag{4.4}$$

Expressions for j_ℓ :

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\ J_{\ell+\frac{1}{2}}(x) = (-1)^\ell x^\ell \left(\frac{d}{xdx}\right)^\ell \frac{\sin x}{x} \ , \tag{4.5}$$

$$\begin{aligned} j_\ell(x) &= \frac{x^\ell}{2^{\ell+1}\,\ell!} \int_{-1}^1 e^{ixs}(1-s^2)^\ell\, \mathrm{d}s \\ &= \frac{2^\ell \ell!}{(2\ell+1)!}\, x^\ell \left(1 - \frac{1}{1!\,(\ell+\frac{3}{2})}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(\ell+\frac{3}{2})(\ell+\frac{5}{2})}\left(\frac{x}{2}\right)^4 - \dots\right) . \end{aligned} \tag{4.6}$$

Recurrence relations:

$$j_{\ell+1} = \frac{\ell}{x} j_\ell - j'_\ell = \frac{2\ell+1}{x} j_\ell - j_{\ell-1}. \tag{4.7}$$

Examples:

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \ ; \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \ ; \\ j_2(x) &= \frac{3\sin x}{x^3} - \frac{3\cos x}{x^2} - \frac{\sin x}{x}. \end{aligned} \tag{4.8}$$

5 Hermite Polynomials $H_n(x)$.

Differential equation:

$$H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0 \ ,$$

or

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(H_n(x)\, e^{-\frac{1}{2}x^2}\Big) + (2n - x^2 + 1)\, H_n(x)\, e^{-\frac{1}{2}x^2} = 0 \ . \tag{5.1}$$

Generating function:

$$\sum_{n=0}^{\infty} H_n(x)\, s^n/n! = e^{-s^2+2sx}. \tag{5.2}$$

Orthogonality:

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2} \mathrm{d}x = 2^n n! \sqrt{\pi}\, \delta_{nm} \tag{5.3}$$

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)/(2^n n!) = \sqrt{\pi}\, \delta(x-y)\, e^{x^2} \ . \tag{5.4}$$

More general:

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, s^n/(2^n n!) = \frac{1}{\sqrt{1-s^2}} \exp\Big(\frac{-s^2(x^2+y^2) + 2sxy}{1-s^2}\Big) \ . \tag{5.5}$$

Expressions for H_n :

$$H_n(-x) = (-1)^n\, H_n(x); \tag{5.6}$$

$$H_n(x) = (-1)^n\, e^{x^2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n e^{-x^2} \ ; \tag{5.7}$$

$$H_n(x) = (-1)^{n/2} n! \sum_{k=0}^{n/2} (-1)^k \frac{(2x)^{2k}}{(2k)!\, (\frac{1}{2}n - k)!} \ , \qquad \text{if } n \text{ even},$$

$$H_n(x) = (\,-1)^{\frac{n-1}{2}} n! \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!\, \Big(\frac{n-1}{2} - k\Big)!} \ , \qquad \text{if } n \text{ odd}. \tag{5.8}$$

Recurrence relations:

$$\frac{\mathrm{d}^m H_n(x)}{\mathrm{d}x^m} = \frac{2^m n!}{(n-m)!}\, H_{n-1}(x) \ , \tag{5.9}$$

$$x\, H_n(x) = \tfrac{1}{2} H_{n+1}(x) + n\, H_{n-1}(x) \ , \tag{5.10}$$

$$H_n(x) = \Big(2x - \frac{\mathrm{d}}{\mathrm{d}x}\Big)\, H_{n-1}(x) \ . \tag{5.11}$$

Examples:

$$H_0(x) = 1 \ , \qquad H_1(x) = 2x \ , \qquad H_2(x) = 4x^2 - 2 \ . \tag{5.12}$$

6 Laguerre Polynomials $L_n(x)$.

Differential equation:

$$x\,L''_n(x) + (1-x)\,L'_n(x) + n\,L_n(x) = 0\ . \tag{6.1}$$

Generating function:[2]

$$\sum_{n=0}^{\infty} L_n(x)\,z^n = \frac{1}{1-z}\,e^{\frac{-xz}{1-z}}\ . \tag{6.2}$$

Orthogonality:

$$\int_0^{\infty} L_n(x)\,L_m(x)\,e^{-x}\,\mathrm{d}x = \delta_{nm}\ . \tag{6.3}$$

Expressions for L_n :

$$\begin{aligned} L_n(x) &= \frac{e^x}{n!}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n (x^n\,e^{-x}) \\ &= \frac{(-1)^n}{n!}\Big(x^n - \frac{n^2}{1!}\,x^{n-1} + \frac{n^2(n-1)^2}{2!}\,x^{n-2} - \ldots + (-1)^n n!\Big)\ . \end{aligned} \tag{6.4}$$

Recurrence relation:

$$\begin{aligned} (1+2n-x)\,L_n - n\,L_{n-1} - (n+1)L_{n+1} = 0\ ; \\ x\,L'_n(x) = n\,L_n(x) - n\,L_{n-1}(x)\ . \end{aligned} \tag{6.5}$$

Examples:

$$\begin{aligned} L_0(x) &= 1\ ; \\ L_1(x) &= 1-x\ ; \\ L_2(x) &= \frac{1}{2!}(x^2-4x+2). \end{aligned} \tag{6.6}$$

[2] It's important to note that sometimes different definitions are used for the Laguerre and Associated Laguerre polynomials, where the Generating Function has the form: $\sum_{n=0}^{\infty} L_n(x)\,z^n/n! = \frac{1}{1-z}\,e^{\frac{-xz}{1-z}}$. In this case the Expressions given for L_n should be multiplied by $n!$.

7 Associated Laguerre Polynomials $L_n^k(x)$.

Differential equation:

$$x\,L_n^{k\,\prime\prime} + (k+1-x)\,L_n^{k\,\prime} + n\,L_n^k = 0\ . \tag{7.1}$$

Generating function:

$$\sum_{n=0}^{\infty} L_n^k(x)\,z^n = \frac{1}{(1-z)^{k+1}}\,e^{\frac{-xz}{1-z}}\ . \tag{7.2}$$

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{L_n^k(x)\,z^n u^k}{k!} = \frac{1}{1-z}\ \exp\Big(\frac{-xz+u}{1-z}\Big)\ . \tag{7.3}$$

Orthogonality:

$$\int_0^{\infty} L_n^k(x)\,L_m^k(x)\,x^k\,e^{-x}\,\mathrm{d}x = \frac{(n+k)!}{n!}\delta_{nm}\ . \tag{7.4}$$

Expressions for L_n^k :

$$L_n^k(x) = (-1)^k\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^k L_{n+k}(x)\ . \tag{7.5}$$

$$L_n^k(x) = \frac{e^x x^{-k}}{n!}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n (x^{n+k}\,e^{-x})\ . \tag{7.6}$$

Recurrence relation:

$$\begin{gathered} L_{n-1}^k\,(x) + L_n^{k-1}\,(x) = L_n^k\,(x)\ ; \\ x\,L_n^{k\,\prime}\,(x) = n\,L_n^k\,(x) - (n+k)L_{n-1}^k\,(x)\ . \end{gathered} \tag{7.7}$$

Examples:

$$\begin{aligned} L_0^k(x) &= 1\ ; \\ L_1^k(x) &= -x+k+1\ ; \\ L_2^k(x) &= \frac{1}{2}\Big[x^2 - 2(k+2)x + (k+1)(k+2)\Big]\ ; \\ L_3^k(x) &= \frac{1}{6}\Big[-x^3 + 3(k+3)x^2 - 3(k+2)(k+3)x + (k+1)(k+2)(k+3)\Big]. \end{aligned} \tag{7.8}$$

8 Tschebyscheff[3] Polynomials $T_n(x)$.

Differential equation:

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,T_n(x) - x\frac{\mathrm{d}}{\mathrm{d}x}\,T_n(x) + n^2\,T_n(x) = 0\ . \tag{8.1}$$

Generating function:

$$\sum_{n=0}^{\infty} T_n(x)\,y^n = \frac{1-xy}{1-2xy+y^2}. \tag{8.2}$$

Symmetry relation:

$$T_n(x) = T_{-n}(x). \tag{8.3}$$

Orthogonality:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\mathrm{d}x = \begin{cases} \frac{1}{2}\pi\delta_{nm} & m,n \neq 0 \\ \pi & m=n=0 \end{cases} \tag{8.4}$$

Expression for T_n:

$$T_n(x) = \cos(n\cos^{-1}x) \tag{8.5}$$

$$T_n(x) = \tfrac{1}{2}\left[\left\{x+i\sqrt{1-x^2}\right\}^n + \left\{x-i\sqrt{1-x^2}\right\}^n\right]. \tag{8.6}$$

Recurrence relation:

$$T_{n+1} - 2x\,T_n(x) + T_{n-1} = 0 \tag{8.7}$$

$$(1-x^2)T_n'(x) = -nx\,T_n(x) + n\,T_{n-1}(x). \tag{8.8}$$

Examples:

$$\begin{aligned} T_0(x) &= 1\ ; \\ T_1(x) &= x\ ; \\ T_2(x) &= 2x^2-1\ ; \\ T_3(x) &= 4x^3-3x. \end{aligned} \tag{8.9}$$

[3]Transliterations Chebyshev and Tchebicheff also occur.

9 Remark.

All of the functions discussed here are special cases of "hypergeometric functions" ${}_mF_n(a_1, a_2, \ldots a_m; b_1, b_2, \ldots b_n; z)$ defined by:

$$ {}_mF_n(a_1, a_2, \ldots a_m; b_1, b_2, \ldots b_n; z) = \sum_{r=p}^{\infty} \frac{(a_1)_r (a_2)_r \ldots (a_m)_r z^r}{(b_1)_r (b_2)_r \ldots (b_n)_r r!}, \tag{9.1} $$

where

$$ (a)_r \equiv \frac{\Gamma(a+r)}{\Gamma(a)}; \qquad r \text{ a positive integer.} \tag{9.2} $$

Differential equations:
$m = n = 1$:

$$ z\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^2 {}_1F_1 + (b - z)\frac{\mathrm{d}}{\mathrm{d}z} {}_1F_1 - a\ {}_1F_1 = 0\ . \tag{9.3} $$

$m = 2, n = 1$:

$$ z(1-z)\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^2 {}_2F_1 + \Big(c - (a+b+1)z\Big)\frac{\mathrm{d}}{\mathrm{d}z} {}_2F_1 - ab\ {}_2F_1 = 0\ . \tag{9.4} $$

We have:

$$ P_\ell(x) = {}_2F_1\left(-\ell, \ell+1; 1; \frac{1-x}{2}\right)\ ; \tag{9.5} $$

$$ P_\ell^m(x) = \frac{(\ell+m)!}{(\ell-m)!}\frac{(1-x^2)^{m/2}}{2^m m!} {}_2F_1\left(m-\ell, m+\ell+1; m+1; \frac{1-x}{2}\right)\ ; \tag{9.6} $$

$$ J_n(x) = \frac{e^{-ix}}{n!}\left(\frac{x}{2}\right)^n {}_1F_1(n+\tfrac{1}{2}; 2n+1; 2ix)\ ; \tag{9.7} $$

$$ H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1(-n; \tfrac{1}{2}; x^2)\ ; \tag{9.8} $$

$$ H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!}{n!} x\, {}_1F_1(-n; \tfrac{3}{2}; x^2)\ ; \tag{9.9} $$

$$ L_n(x) = {}_1F_1(-n; 1; x)\ ; \tag{9.10} $$

$$ L_n^k(x) = \frac{\Gamma(n+k+1)}{n!\Gamma(k+1)} {}_1F_1(-n; k+1; x)\ ; \tag{9.11} $$

$$ T_n(x) = {}_2F_1\left(-n, n; \tfrac{1}{2}; \frac{1-x}{2}\right)\ . \tag{9.12} $$

References

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