Κβαντική Μηχανική Ι.
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Coherent States

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Consider a coherent state $|a\rangle$ associated to the complex number $a \in C$

$$\hat{a}\ |a\rangle = a\ |a\rangle \tag{1}$$

We would like to find its "wave-function" in the position representation, namely, $\psi_a(x) \equiv \langle x|a\rangle$.

We start by taking the inner product of (1) with $\langle x|$ to obtain

$$\langle x|\ \hat{a}\ |a\rangle = a\ \langle x|a\rangle = a\psi_a(x) \tag{2}$$

Remember that in the position representation

$$|\psi\rangle \to \langle x|\psi\rangle = \psi(x)\ \ ,\ \ \hat{p}|\psi\rangle \to \langle x|\hat{p}|\psi\rangle = -i\frac{\partial}{\partial x}\psi(x)\ \ ,\ \ \hat{x}|\psi\rangle \to \langle x|\hat{x}|\psi\rangle = x\psi(x) \tag{3}$$

On the other hand

$$\langle x|\hat{a}|a\rangle = \frac{1}{\sqrt{2}}\langle x|\hat{x}+i\hat{p}|a\rangle = \frac{1}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\langle x|a\rangle = \frac{1}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\psi_a(x) \tag{4}$$

Together with (**??**) it implies

$$x\psi_a(x) + \psi_a(x)' = \sqrt{2}a\psi_a(x) \tag{5}$$

whose most general solution is

$$\psi_a(x) = C\ e^{-\frac{x^2}{2}+\sqrt{2}ax} = C'\ e^{-\frac{1}{2}(x-\sqrt{2}a_1)^2+i\sqrt{2}a_2x}\ \ ,\ \ C' = Ce^{a_1^2} \tag{6}$$

where we have written the complex number $a = a_1 + ia_2$ with $a_1, a_2 \in R$. We also normalize it

$$1 = \int_{-\infty}^{+\infty} dx\ |\psi_a(x)|^2 = |C'|^2 \int_{-\infty}^{+\infty} dx\ e^{-(x-\sqrt{2}a_1)^2} = |C'|^2 \int_{-\infty}^{+\infty} dy\ e^{-y^2} = \sqrt{\pi}|C'|^2 \to C' = \frac{1}{\pi^{\frac{1}{4}}} \tag{7}$$

where in the third step we changed integration variables from x to $y = x - \sqrt{2}a_2$.

The probability density associated with this wave-function is

$$P_a(x) = |\psi_a(x)|^2 = \frac{1}{\sqrt{\pi}}e^{-(x-\sqrt{2}a_1)^2} \tag{8}$$

and is therefore a Gaussian centered around $x = \sqrt{2}a_1$. Therefore the mean value of the position is

$$\langle x\rangle = \langle a|\hat{x}|a\rangle = \int_{-\infty}^{\infty} dx\, x|\psi_a(x)|^2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, x\, e^{-(x-\sqrt{2}a_1)^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dy\, (y+\sqrt{2}a_1)\, e^{-y^2} = \sqrt{2}a_1 \tag{9}$$

where in the third step we changed integration variables to $y = x - \sqrt{2}a_1$ and in the last we used that $\int_{-\infty}^{\infty} dy\ y\ e^{-y^2} = 0$ due to symmetry.

We can also compute the expectation value of the momentum in this state

$$\langle p\rangle \equiv \int_{-\infty}^{\infty} dx\ \psi_a^* \left(-i\frac{\partial}{\partial x}\right)\psi_a = \tag{10}$$

$$= -\frac{i}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}(x-\sqrt{2}a_1)^2 - i\sqrt{2}a_2 x}\left(-(x-\sqrt{2}a_1) + i\sqrt{2}a_2\right) e^{-\frac{1}{2}(x-\sqrt{2}a_1)^2 + i\sqrt{2}a_2 x} =$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dy\ \left(iy + \sqrt{2}a_2\right) e^{-y^2} = \sqrt{2}a_2$$

Therefore the state has semi-classical "position" $\sqrt{2}a_1$ and semiclassical "momentum" $\sqrt{2}a_2$ or

$$a = \frac{\langle x\rangle + i\langle p\rangle}{\sqrt{2}} \tag{11}$$

When we evolve in time with the harmonic oscillator Hamiltonian $a \to a^{e^{-it}}$ so that

$$a_1(t) = a_1 \cos t\ \ ,\ \ a_2(t) = -a_2 \sin t\,. \tag{12}$$

Therefore the semiclassical position and momentum move according to the classical equations of motion.